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EQUATIONS OF LUR'E IN HILBERT SPACE AND THEIR SOLVABILITY

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A novel proof of the solvability of the Lur'e equations which appear in the theory of absolute stability is presented. This proof makes it possible to generalize the results to the Hilbert spaces and unbounded operators.

Let the following real matrices be given: A and P are $n \times n$ matrices, B and Q are $n \times m$ matrices and R is an $m \times m$ matrix; P and R are symmetric ones. We regard as the Lur'e equations, the equations with respect to the real symmetric $n \times m$ matrix M and $n \times m$ matrix L of the form

$$MA + A^*M = -P + LL^*, \quad LK = MB + Q, \quad K^*K \equiv R \tag{0.1}$$

The equations (0.1) with $m = 1$ were introduced in a slightly different form by Lur'e in connection with the study of the problem of absolute stability [1]. The importance of these equations here lies in the fact that as soon as their solutions exists, so does the global Liapunov function of the initial nonlinear system.

Equations of the type (0.1) for an arbitrary value of m also called the generalized Lur'e equations, were introduced in [2] in connection with the generalized problem of absolute stability containing many nonlinearities. The sufficient conditions of solvability of Eqs. (0.1) in their closed form, for $m = 1$, A — Hurwitz matrix, $R > 0$ and $R = 0$, $P < 0$, were first given in [3]. These conditions were generalized further [3 — 7] to the case of an arbitrary $m \geqslant 1$, $R \geqslant 0$ and an arbitrary matrix A, and sharpened to the necessary and sufficient conditions: $\Pi(\omega) \geqslant 0$, $\omega \in (-\infty, +\infty)$ where

$$\Pi(\omega) = R + 2\mathrm{Re}\,(Q^*(j\omega E - A)^{-1}B) + B^*(-j\omega E - A^*)^{-1}P(j\omega E - A)^{-1}B \tag{0.2}$$

the asterisk denotes the transpose of the matrix, E is a unit $n \times n$ matrix and $j = \sqrt{-1}$). The case $R \geqslant 0$ was also investigated in [8].

We note that the connection between the solution of (0.1) and the solution of the variational problem was first discovered by Popov (see [7] who showed that the solvability of (0.1) implies the existence of a solution of the corresponding variational problem. Below we establish the converse relation with the help of an idea of Lyons concerning the unlinking of the Hamiltonian systems [9].

1. Novel proof of the solvability of the Lur'e equations in the case of $R > 0$. The proof is based on the following theorem (the assertion 1° of the theorem is already known, see [10]).

Theorem 1. Let A be a Hurwitz matrix and let the following frequency inequality hold for some $\varepsilon > 0$:

$$\Pi(\omega) \geqslant \varepsilon E \tag{1.1}$$

Then the assertions 1° and 2° both hold.

1°. There exists a solution of the variational problem of the minimum of the functional $I_h[u(t)]$ (h is an n-vector and u is an m-vector)

$$I_n[u(t)] = \int_0^\infty \mu(y(h,u)(t), y(h,u)(t);\ u(t), u(t))\,dt \tag{1.2}$$

for all $u(t)$, $t \in (0, \infty)$ satisfying the inequality

$$\rho^2(u) \equiv \int_0^\infty \|u(t)\|_m^2 dt < \infty \tag{1.3}$$

$$(\|u(t)\|_m^2 = \langle u(t),\ u(t)\rangle_m = \sum_{i=1}^m u_i^2(t), \quad u = (u_1, \ldots, u_m))$$

where

$$\mu(x, y;\ v, u) = \langle Py,\ x\rangle_n + \langle x,\ Qv\rangle_n + \langle y,\ Qu\rangle_n + \langle Ru,\ v\rangle_m \tag{1.4}$$

$$y(h,u)(t) = e^{At}h + \int_0^t e^{A(t-\tau)} u(\tau)\,d\tau$$

The solution is unique within the values on the set of the t-axis of zero measure.

2°. The symmetric matrix M defined by the expression

$$\langle h_1, Mh_2\rangle = \langle Mh_1, h_2\rangle = \int_0^\infty \mu(y(h_1, u_1^\circ),\ y(h_2, u_2^\circ);\ u_1^\circ, u_2^\circ)\,dt \tag{1.5}$$

where $u_i^\circ(t)$ is the solution of the variational problem formulated above with $h = h_i$ ($i = 1, 2$), exists and satisfies the relations (0.1). From now on we shall omit the dimension indices at the scalar products and norms, in the cases when their dimensions are obvious.

Proof. 1°. Since A is a Hurwitz matrix, the functional $I_h[u]$ is defined for all functions $u(t)$ satisfying (1.3), and is continuous on the norm (1.3). The functional I_h can be written in the following form:

$$I_h[u] = \pi_0[u(t), u(t)] - 2L_h[u(t)] + I_h^\circ \tag{1.6}$$

$$L_h[u] = -\int_0^\infty [\langle Py(u),\ y(h)\rangle + \langle y(h),\ Qy(u)\rangle]\,dt$$

$$y(u)(t) = y(\theta, u)(t), \quad y(h)(t) = e^{At}h$$

$$\pi_0[u, v] = \int_0^\infty \mu(y(u),\ y(v);\ u, v)\,dt, \quad I_h^\circ = I_h[\theta_m(t)]$$

($\theta_m(t)$ is an identically zero function belonging to the domain of definition of the functional).

The form $\pi_0[u, v]$ is continuous, symmetric and bilinear. For the functions $u(t)$ satisfying the condition (1.3) we have, by virtue of the Parseval equation and condition (1.1),

$$\pi_0[u(t), u(t)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \langle \Pi(\omega) f_u(\omega), f_u(\omega)\rangle\,d\omega \geqslant \tag{1.7}$$

$$\varepsilon\int_0^\infty \|u(t)\|^2 dt = \varepsilon\rho^2(u), \quad f_u(\omega) \equiv \int_0^\infty u(t)\,e^{j\omega t}\,dt$$

(the form $\pi_0[u, v]$ is coercive [9]). From this it follows [9] that the extremal element $u^\circ(t)$ of the functional (1.2) exists and is unique under the condition (1.3). The uniqueness of the extremal element $u^\circ(t)$ is understood in the sense of the norm of $\rho(u)$.

2°. From [9] it also follows that the extremal function $u^\circ(t)$ must satisfy the equation

$$\pi_0[u^\circ(t), u(t)] = L_h[u(t)] \tag{1.8}$$

for all $u(t)$ satisfying the condition (1.3). Taking into account (1.4) and (1.6), we can transform the above equation to the form

$$\int_0^\infty [\langle Py^\circ, y\rangle + \langle y^\circ, Qu\rangle + \langle y, Qu^\circ\rangle + \langle Ru^\circ, u\rangle]\, dt = 0 \tag{1.9}$$

$$(y^\circ(t) = y(h, u^\circ)(t),\ y(t) = y(u)(t) = y(\theta, u)(t))$$

Conversely, only the extremal element of the functional I_h can be a solution $u^\circ(t)$ of (1.9) with condition (1.7) (with the accuracy within the values on the set of measure zero).

Next we define the continuous function $\Psi(t)$

$$\Psi(t) = \int_t^\infty e^{-A^*(t-\tau)}(Py^\circ(\tau) + Qu^\circ(\tau))\, d\tau \tag{1.10}$$

The function $\Psi(t)$ obviously satisfies the relations (1.11)

$$\frac{d}{dt}\Psi = -A^*\Psi - Py^\circ(t) - Qu^\circ(t) \tag{1.11}$$

$$\int_0^\infty \|\Psi(t)\|^2 dt < \infty, \quad \lim_{t\to\infty}\|\Psi(t)\| = 0 \tag{1.12}$$

By virtue of (1.6), (1.9) — (1.12), we have

$$0 = \int_0^\infty \left\langle \frac{d}{dt}\Psi + A^*\Psi + Py^\circ + Qu^\circ, y(u)\right\rangle dt = \int_0^\infty \left\langle \frac{d}{dt}\Psi + A^*\Psi, y(u)\right\rangle dt - \tag{1.13}$$

$$\int_0^\infty [\langle y^\circ, Qu\rangle + \langle Ru^\circ, u\rangle]\, dt = \int_0^\infty \left\langle -\frac{d}{dt}y(u) + Ay(u), \Psi\right\rangle dt -$$

$$\int_0^\infty \langle Q^*y^\circ + Ru^\circ, u\rangle\, dt = \int_0^\infty \langle -Bu, \Psi\rangle\, dt - \int_0^\infty \langle Q^*y^\circ + Ru^\circ, u\rangle\, dt =$$

$$-\int_0^\infty \langle B^*\Psi + Q^*y^\circ + Ru^\circ, u\rangle\, dt$$

Since $u(t)$ in these equations is an arbitrary function satisfying (1.3), we have the relation $B^*\Psi(t) = -Ru^\circ(t) - Q^*y^\circ(t)$. By virtue of the condition (1.1) $R > 0$, therefore (1.13) can be written in the form

$$u^\circ(t) = -R^{-1}(B^*\Psi(t) + Q^*y^\circ(t)) \tag{1.14}$$

The above relation implies, in particular, the continuous character of $u^\circ(t)$.

Let us consider the mapping which places the n-vector h in correspondence with the function Ψ. The mapping is linear by virtue of (1.9) and (1.10). It follows that there exists a $n \times n$ matrix M such, that $Mh = \Psi(0)$. Moreover, it turns out that the matrix M satisfies the equation

$$My^\circ(t) = \Psi(t), \quad t \geqslant 0 \tag{1.15}$$

The proof of the relation (1.15) is based on the concepts given in Ch. 3 of the monograph

[9]. Consider the functional

$$I_h^s[u] = \int_s^\infty \mu(y_s(h, u), y_s(h, u); u, u)\,dt, \quad s \geqslant 0 \tag{1.16}$$

$$y_s(h, u)(t) = e^{A(t-s)} h + \int_s^t e^{A(t-\tau)} u(\tau)\,d\tau$$

Repeating the previous arguments for this functional, denoting by $u_s^\circ(h, t)$ the extremal element of the corresponding variational problem and by $y_s^\circ(h, t)$ and $\Psi_s(h, t)$ the functions of the form

$$y_s^\circ(h, t) = y_s(h, u_s^\circ)(t), \quad \Psi_s(h, t) = \int_t^\infty e^{-A^*(t-\tau)} \times$$

$$[Py_s^\circ(h, \tau) + Qu_s^\circ(h, \tau)]\,d\tau, \quad t \geqslant s$$

we find that there exists a matrix M_s satisfying the relation

$$M_s y_s^\circ(g, s) = \Psi_s(g, s) \quad (g \in R^n, \ s \geqslant 0) \tag{1.17}$$

Let us take, as g in (1.17), the vector $g = y^\circ(h, s)$ which represents the value of the function $y^\circ(t)$ of the initial variational problem $(s = 0)$ at the instant $t = s$. Then by virtue of the stationary character of the problem, we have

$$y_s^\circ(g, t) = y_0^\circ(g, t - s) = y^\circ(g, t - s), \quad \Psi_s(g, t) = \Psi_0(g, t - s)$$

and consequently the operator M_s in (1.17) is independent of s. Further, by virtue of the principle of optimality, the fact that the functions $u^\circ(t)$ and $u_s^\circ(t)$ are extremal, it follows

$$y_s^\circ(g, t) = y_0^\circ(h, t), \quad \Psi_s(g, t) = \Psi_0(h, t) \quad g = y_0^\circ(h, s), \quad h \in R^n, \quad t \geqslant s$$

which, together with (1.17) (taking into account the fact that M_s is independent of s, and s is arbitrary), yield (1.15).

The relations (1.4), (1.11), (1.12), (1.14) and (1.15) in turn yield, for any $h_1, h_2 \in R^n$, the following sequence of equations:

$$\langle Mh_1, h_2\rangle = \langle \Psi(h_1, 0), y^\circ(h_2, 0)\rangle = -\int_0^\infty \left[\left\langle \frac{d}{dt}\Psi(h_1, t), y^\circ(h_2, t)\right\rangle + \right. \tag{1.18}$$

$$\left.\left\langle \Psi(h_1, t), \frac{d}{dt} y^\circ(h_2, t)\right\rangle\right] dt = \int_0^\infty \mu(y^\circ(h_1, t), y^\circ(h_2, t); u^\circ(h_1, t)$$

$$u^\circ(h_2, t))dt = \langle y^\circ(h_1, 0), \Psi(h_2, 0)\rangle = \langle h_1, Mh_2\rangle$$

i.e. the relation (1.5). Finally from (1.11), (1.14), (1.4) and (1.15) follows

$$\theta_n = M\frac{d}{dt} y^\circ(t) - \frac{d}{dt}\Psi(t) = M(Ay^\circ(t) + Bu^\circ(t) + A^*\Psi(t) +$$

$$Py^\circ(t) + Qu^\circ(t) = (MA + A^*M)y^\circ(t) + (MB + Q)u^\circ(t) + Py^\circ(t)$$

from which, with (1.5) and (1.14) taken into account, we obtain

$$Sy^\circ(h, t) = \theta_n, \quad S = MA + A^*M + P - (MB + Q)R^{-1}(MB + Q)^* \tag{1.19}$$

Going in (1.19) to the limit as $t \to 0$ and assuming that h is an arbitrary n-vector, we find that S is an $n \times n$ zero matrix. This is equivalent, for $R > 0$, to the relation (0.1), which completes the proof.

Theorem 2. Let such $m \times n$ matrix F exist for the matrices A and B where A has no spectrum on the imaginary axis, that $A_1 = A + BF$ is a Hurwitz matrix. Then a symmetric matrix M satisfying (0. 1) will exist provided that the relation

$$\Pi(\omega) \geqslant \varepsilon (E - B^* (-j\omega E - A^*)^{-1} F^*)(E - F(j\omega E - A)^{-1} B) \qquad (1.20)$$

holds for some $\varepsilon > 0$.

Proof. We note that

$$\langle \Pi(\omega; A_1, B, P_1, Q_1, R) f_{u_1}, f_{u_1} \rangle = \langle \Pi(\omega; A, B, P, Q, R) f_u, f_u \rangle$$
$$(u_1(t) = u(t) - Fy(t), \quad P_1 = P + F^*RF + F^*Q^* + QF, \quad Q_1 = Q + F^*R)$$

Therefore, when the condition (1. 20) holds for the matrices A_1, B, P_1, Q_1 and R , Theorem 1 is true and a matrix M will exist satisfying Eqs. (0. 1) for the altered values of the matrices appearing in the expression. Passing in the relations (0. 1) to the initial matrices A, P and Q, we find that M also satisfies the initial relations (0. 1).

Corollary. Let a pair of matrices (A, B) be fully controllable [7]. Let also the matrix A have no spectrum on the imaginary axis and the condition (1. 1) hold. Then a symmetric matrix M exists satisfying the relations (0. 1).

In accordance with [7], if the pair (A, B) is fully controllable, then the matrix F from Theorem 2 will exist. Moreover we note that $\sup_\omega \| F(j\omega E - A)^{-1} B \| < \infty$, and this implies that the above assertion follows from Theorem 2.

2. Sufficient conditions of solvability of the Lur'e equations in a Hilbert space. The proof of Theorem 1 admits a generalization to arbitrary Hilbert spaces and to the case of an unbounded operator A. Such generalization is useful in studying nonlocal stability and instability of dynamic systems described by partial differential equations.

In what follows, we shall use the following notation. If H is a Hilbert space over the field of real numbers, then $\langle . , . \rangle_H$ is a scalar product in this space; θ_H denotes the zero element of H; V^* is a space dual [11] to V; $\langle f, g \rangle$, $g \in V$, $f \in V^*$ is the value of the functional f on the element g; $T: H_1 \to H_2$ is a linear operator acting from H_1 to H_2; $\theta(H_1 \to H_2)$ is a zero mapping $H_1 \to H_2$ (or its equivalence class in the sense of the corresponding norm); $L^2(\tau, T; H)$, $\tau < T$ is a Hilbert space of absolutely square integrable mappings $(\tau, T) \to H$ with a uniquely defined scalar product and $L^2(\tau; H) = L^2(\tau, \infty; H)$. We shall also make use of the Sobolev type [9] space $W(\tau, T; V)$, defined as follows:

$$W(\tau, T; V) = \left\{ y(t) \mid y(t) \in L^2(\tau, T; V), \frac{d}{dt} \in L^2(\tau, T; V^*) \right\}$$

$$\| y(t) \|_W^2 = \int_\tau^T \left(\| y(t) \|_V^2 + \left\| \frac{d}{dt} y(t) \right\|_{V^*}^2 \right) dt$$

So, let H, V and U be Hilbert spaces over the field of real numbers [11], with the inclusion $V \subset H = H^* \subset V^*$, the imbedding $V \to H$ is continuous and the set of elements of the space V is dense everywhere in H. (This, in particular, implies that $\langle f, g \rangle = \langle f, g \rangle_H$ if $f \in H$ and $g \in H$.) Let A be a continuous linear operator $V \to V^*$, closed in the space H , the domain of definition $D(A)$ of which is dense in V.

We note that $D(A)$ is a set of elements h of the space V satisfying the condition $Ah \in H$ [9, 11]. Consequently the operator A will be unbounded (generally speaking) on H. The operator A becomes bounded if $V = H$.

Let us consider a linear evolutionary equation [9] of the form

$$\frac{d}{dt} y(t) = Ay(t) + f(t) \tag{2.1}$$

We shall say, in accordance with [9], that a solution $y(t)$ of (2.1) on some interval (τ, T) is a function of the space $W(\tau, T; V)$ such, that for any function $\xi(t), \xi \in V$ smooth and finite on the interval (τ, T), the following relation holds:

$$\int^{T} \left[\left\langle y(t), \frac{d}{dt}\xi(t) \right\rangle + \langle Ay(t), \xi(t)\rangle + \langle f(t), \xi(t)\rangle \right] dt = 0 \tag{2.2}$$

Assumption 1. For any $h \in H$, $T > 0$ and $f(t) \in L^2(0, T; V^*)$ there exists a unique solution $y(t)$ of (2.1) satisfying the initial condition

$$y(0) = h \tag{2.3}$$

The solution depends continuously on the initial values of $f(t)$ and h in the sense, that $(h, f) \to y(t)$ is a continuous mapping of the space $H \times L^2(0, T; V^*)$ onto the space $W(0, T; V)$.

Assumption 2. For a given function $g(t) \in L^2(0; H)$ a H-continuous function $\Psi(t) \in W(0; V)$ exists and is unique. The latter function has values in the space V, and satisfies (in the sense of Eq. (2.2) in which $T = \infty$, $f = g$ and the operator A is replaced by $[-A^*]$) the equation

$$\frac{d}{dt}\Psi = -A^*\Psi + g(t) \tag{2.4}$$

where A^* is the operator $V \to V^*$ conjugate to A [11].

Note 1. A theorem exists which states that any function belonging to the space $W(0, T; V)$ which is modified on the set of zero measure in the appropriate manner, will be a continuous function of $[0, T] \to H$ [9]. Therefore Eq. (2.3) is meaningful and it also follows that the function $\Psi(t) \in W(0; V)$, representing the continuous mapping $[0, \infty] \to H$, has a zero limit θ_H as $t \to \infty$ [9].

Definition 1. We say that the operator A is L^2.-stable if the Assumption 1 holds for this operator when $T = \infty$. Let B be a linear bounded operator $U \to V^*$. We denote by $y(h, u)(t)$ the solution of the equation

$$dy/dt = Ay + Bu(t), \quad u(t) \in L^2(0; U) \tag{2.5}$$

with the initial condition (2.3). We denote this solution by $y(u)(t)$ when $h = \theta_H$, and by $y(h)(t)$ when $u(t) = \theta(R^1 \to U)$.

Assumption 3. There exists a set V_1, $D(A) \subset V_1 \subset V$ such that if $h \in V_1$ is a function continuous in t, then $y(h, u)(t)$ is a continuous function of $(0, \infty) \to V$. We shall now formulate a theorem which, in the finite-dimensional case, becomes analogous to Theorem 1.

Theorem 3. Let A be an L^2-stable operator and the Assumptions 2 and 3 hold. Let also P, Q, R and B be linear bounded operators $H \to H$, $U \to H$, $U \to U$ and $U \to V^*$, respectively, and let the following inequality hold for some $\varepsilon > 0$ for all $u(t) \in L^2(0; U)$:

$$\int_0^\infty [\langle Ru(t), u(t)\rangle_U + 2\langle y(u)(t), Qu(t)\rangle_H + \tag{2.6}$$

$$\langle Py(u)(t), y(u)(t)\rangle_H]\, dt \geqslant \varepsilon \int_0^\infty \| u(t) \|_U^2\, dt$$

Then the assertions 1° and 2° are valid.

1°. A function $u^\circ(t) \in L^2(0; U)$ satisfying the equation

$$I_h[u^\circ(t)] = \inf_{u(t)\in L^2} I_h[u(t)] \tag{2.7}$$

$$I_h[u(t)] \equiv \int_0^\infty \mu(y(h, u)(t), y(h, u)(t); u(t), u(t))\,dt$$

$$\mu(x, y; v, u) \equiv \langle Py, x\rangle_H + \langle x, Qv\rangle_H + \langle y, Qu\rangle_H + \langle Ru, v\rangle_U$$

exists for any $h \in H$. Since $u^\circ(t)$ is an element of the space $L^2(0; U)$, it can be defined uniquely.

2°. A linear bounded operator $M : H \to V$ exists satisfying, for any $h_i \in H$, the relation (1.5) where $u_i^\circ(t)$ is the solution of the variational problem (2.7) with $h = h_i$, and also satisfying the relation

$$\langle(M^*A + A^*M + P)\xi, \eta\rangle = \langle LL^*\xi, \eta\rangle, \quad L = (M^*B + Q)K^{-1}, \quad K^*K = R; \ \xi, \eta \in V \tag{2.8}$$

where M^* is the operator $V^* \to H$ conjugate to M [11]. (The operator M^* acting from V^* to H is defined by the equation $\langle M^*f, h\rangle = \langle f, Mh\rangle$ which holds for all $f \in V^*$ and $h \in V$.)

The relations (2.8) represent a generalized form of the relations (0.1). In place of the vector spaces R^n, R^m and of the matrices, we have the Hilbert spaces H, V and U and the linear operators acting in these spaces (here the operator A is an unbounded operator in H). The latter circumstance enables us to utilize the relations (2.8) during the investigations of nonlinear distributed systems in the manner, in which Eqs. (0.1) were used (by Iakubovich and his coworkers) in studying the systems described in terms of the ordinary differential equations. Therefore Eqs. (2.8) can be regarded as the Lur'e equations in Hilbert spaces.

Note 2. Theorem 2 gives the sufficient conditions of solvability of the Lur'e equations, provided that A is a stable operator and R is a positive definite operator. The method discussed here was used in [12] to study the case $R \geqslant 0$, but the operators A in that case were bounded.

Note 3. In the case when the Fourier transform $f_y(\omega)$ of the function $\{y(u)(t)$ when $t > 0$ and θ_H when $t < 0\}$ is connected with the Fourier transform $f_u(\omega)$ of the function $\{u(t)$ when $t \geqslant 0$ and θ_u when $t < 0\}$ by the relation

$$f_y(\omega) = (j\omega E - A)^{-1}Bf_u(\omega) \tag{2.9}$$

(which will hold when $Bu \in H$, as well as in a number of problems, where $Bu \in V^*$, e.g. in the investigation of controlled systems described by partial differential equations with the control appearing in the boundary conditions [9]), the condition (2.7) can be written in the following frequency form:

$$\langle\Pi(\omega)u, u\rangle \geqslant \varepsilon\|u\|^2, \quad \Pi(\omega) \equiv R + Q^*(j\omega E - A)^{-1}B + B^*(-j\omega E - A^*)^{-1}Q + B^*(-j\omega E - A^*)^{-1}P(j\omega E - A)^{-1}B \tag{2.10}$$

When the operator B has its left inverse B_-, the condition (2.6) can be replaced by

$$\langle[(-j\omega E - A^*)B_-^*(R - \varepsilon E)B_-(j\omega E - A) + 2\mathrm{Re}\,QB_-(j\omega E - A) + P]f_y(\omega), f_y(\omega)\rangle \geqslant 0, \quad \omega \in (-\infty, +\infty) \tag{2.11}$$

In contrast to the condition (2.10), (2.11) contains the operator A itself instead of its resolvent. In the case of differential operators A this helps to obtain the relations between the parameters for which the condition (2.6) holds, while investigating specific partial differential equations.

Proof of Theorem 3. 1°. We write the functional $I_h[u]$, just as in the finite-dimensional case, in the form (1.6). But now $\pi_0[u, u]$ and $L_h[u]$ represent the functionals $U \times U \to R^1$ and $H \times U \to R^1$ in which μ, P Q and R are the operators from the condition of Theorem 3, while $y(u)(t)$ and $y(h)(t)$ are the corresponding solutions of (2.5). The form $\pi_0[u, v]$: $L^2(0, U) \times L^2(0, U) \to R^1$ is a continuous and symmetric one; when the condition (2.6) holds, it also becomes coercive [9]. This implies, by virtue of Theorem 1.1 of ch. 1 of [9], the existence and uniqueness of the element $u^\circ(t) \in L^2(0; U)$ satisfying the relation (2.7).

2°. We introduce the function $\Psi(t) \in W(0; V)$, having defined it in accordance with the Assumption 2 as the unique solution of the equation

$$\frac{d}{dt}\Psi(t) = -A^*\Psi(t) - Py^\circ(t) - Qu^\circ(t), \quad y^\circ(t) = y(h, u^\circ)(t) \tag{2.12}$$

By virtue of (2.12) and Assumptions 1 and 2, the operations in (1.13) remain valid here; moreover B^* and Q^* are the operators $V \to U$ and $H \to U$ conjugate to the operators B and Q. Thus we arrive at the relation

$$u^\circ(t) = -R^{-1}(B^*\Psi(t) + Q^*y^\circ(t)) \tag{2.13}$$

Further, the functions $y^\circ(t)$ and $\Psi(t)$, and by virtue of (2.13) the function $u^\circ(t)$, can all be assumed continuous on the interval $[0, \infty]$ (after a possible change in their values on the set of zero measure), therefore a mapping $M: h \to \Psi(0)$, exists which will be linear for reasons analogous to those in the finite-dimensional case. In fact, the mapping M is a product of the linear mappings $H \ni h \to (u^\circ(t)$ and $y^\circ(t)) \to \Psi(0) \in V$. The linearity of the first mapping follows from the infinite-dimensional analog of (1.9). The existence of the second mapping is guaranteed by the Assumption 2.

Let us consider, for an arbitrary $s \geqslant 0$, the functional $Ig^s(u)$ of the form (1.16) in which $u(t) \in L^2(s; U)$ and $y_s(g, u)(t) \in W(s; V)$ is a solution of (2.5) in the interval (s, ∞) satisfying the initial condition $y_s(g, u)(s) = g \in H$. (Such a solution exists and is unique by virtue of the Assumption 1 and the time independence of the operators A and B). Since the operators A, B, P Q and R are stationary, the extremal element $u_s^\circ(g, t)$ of this functional can be obtained from the extremal element $u^\circ(g, t)$ of the functional (2.7) by displacing it in time $u_s^\circ(g, t) = u^\circ(g, t - s)$. This also implies that $y_s^\circ(g, u_s^\circ)(t) = y^\circ(g, u^\circ)(t - s)$.

Let us now introduce the function $\Psi_s(g, t) \in W(s; V)$ as a solution of the equation

$$\frac{d}{dt}\Psi_s = -A^*\Psi_s - Py_s^\circ(t) - Qu_s^\circ(t), \quad t > s$$

$$(\Psi_s \in W(s, V),\ y_s^\circ(t) = y_s^\circ(g, u_s^\circ)(t))$$

Clearly, $\Psi_0(t) = \Psi(t)$ and $\Psi_s(g, t) = \Psi_0(g, t - s)$.

From the above relations it follows that the linear operator which puts the elements $g \in H$ and $\Psi_s(g, s)$ in one-to-one correspondence, is the operator M defined above. Using now the arguments employed in Theorem 1 and based on the principle of optimality (the dimensionality of the spaces does not enter this argument), we obtain

$$My^\circ(t) = \Psi(t), \quad t \geqslant 0 \tag{2.14}$$

Further, in accordance with the definition of the functions $y^\circ(t)$ and $\Psi(t)$, all transformations in (1.18) remain valid and the operator M will continue to satisfy the relations (1.5) in the space H. In particular, we find that $\langle Mh_1, h_2\rangle = \langle h_1, Mh_2\rangle$ for any $h_i \in H$. From this it follows that [11] the operator M, as operator $H \to H$, is Hermitian.

On the other hand, in accordance with the Assumption 2 we have $Mh = \Psi(0) \in V$, so that M will be a linear mapping $H \to V$. The closed graph theorem [11] (since $D(M) = H$ and the mapping M is closed) now shows that the mapping M is continuous and, as the mapping $H \to V$. The theorem also implies the existence of the continuous operator $M^*: V^* \to H$ conjugate to $M: H \to V$. We note that $M^*h = Mh$ for $h \in H$.

Let us now consider the relation

$$\int_0^T \left[\left\langle \frac{d}{dt}\Psi + A^*\Psi + Py^\circ + Qu^\circ, \xi\right\rangle - \left\langle \frac{d}{dt}y^\circ - Ay^\circ - Bu^\circ, M\xi\right\rangle\right] dt = 0 \tag{2.15}$$

$$(\xi(t) \in W(0, T; V))$$

The validity of (2.15) follows from the definitions of the functions $y^\circ(t)$ and $\Psi(t)$ and of the operator M, and together with (2.13) and (2.14) it yields

$$\int_0^T \langle Sy^\circ(t), \xi(t)\rangle\, dt = 0, \quad \forall \xi(t) \in W(0, T; V) \tag{2.16}$$

$$S \equiv M^*A + A^*M + P + (M^*B + Q)R^{-1}(B^*M + Q^*)$$

The operator S is continuous on $V \to V^*$. Therefore from (2.16) (with the Assumption 3 taken into account) it follows that $Sh = \theta$ if $h \in V_1$. Since the element $h \in V_1$ is arbitrary and V_1 is dense in V, this becomes equivalent to the relation (2.8), and this completes the proof of Theorem 3.

Note 4. From the proof of the theorem it follows that Assumption 2 can be relaxed. It is sufficient to require that it only holds for the functions $g(t)$ of the form $Py^\circ(t) + Qu^\circ(t)$.

Theorem 4. Assume that the Assumptions 1 — 3 hold and a linear bounded operator $F: H \to U$ exists which satisfies the following conditions: operator $A_1 = A + BF$ is L^2-stable, and for some $\varepsilon > 0$ the following inequality is satisfied

$$\int_0^\infty [\langle Rv(t), v(t)\rangle + 2\langle y(v)(t), Qv(t)\rangle + \langle Py(v)(t), y(v)(t)\rangle]\, dt \geqslant \tag{2.17}$$

$$\varepsilon \int_0^\infty \| v(t) - Fy(v)(t)\|^2 dt, \quad v(t) - Fy(v)(t) \in L^2(0, U)$$

Then a continuous linear operator $M: H \to V$, satisfying the conditions (2.8) will exist. The proof is similar to that of Theorem 2, and is carried out by passing to Theorem 3 with help of the substitution $u = v - Fy$.

If the relation (2.9) holds for the operators A_1 and B, the inequality (2.17) can be written in the following frequency form:

$$\langle \Pi(\omega)v, v\rangle \geqslant \varepsilon \| v - F(j\omega E - A)^{-1}Bv\|^2$$

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ON THE CONSTRUCTION OF GENERAL SOLUTIONS OF THE ELASTICITY THEORY EQUATIONS FOR TRANSVERSELY ISOTROPIC INHOMOGENEOUS BODIES

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The solution is presented for the three-dimensional problem of the theory of elasticity of transversely isotropic elastic bodies, where the elastic characteristics vary arbitrarily along the axis of symmetry of the elastic properies of the medium. The solution is written in orthogonal curvilinear cylindrical coordinates and is represented by using two independent functions. The question of separation of the boundary conditions in the plane of isotropy is examined.

A number of investigations, which examine primarily the equilibrium of